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Linear System Input-Order Reduction by Hankel Norm Maximization

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I. Introduction

CONTROL blending and disturbance direction identification are two examples of linear system input-dimension reduction problems. In control blending problems, system input dimension is reduced to facilitate control design and implementation. Typically, the system has dynamically redundant actuators that are to be mapped to a smaller set of generalized controls and an associated control mapping.

In disturbance decoupling problems where the disturbance is meant to model neglected higher-order or nonlinear dynamics, determination of the direction from first principles is not always practical or possible. When the disturbance direction is found empirically, typically several directions are found, each one associated with a different operating point, and a suitable representative direction must be chosen.[1] The problem is complicated further when the rank of the disturbance map is not known, that is, when it is not clear how many directions should be chosen from the empirically derived set.

One approach to model input reduction, appealing at first, is to inspect, pairwise, the angle between each input direction. If all the angles are small it might seem reasonable to choose any one direction or an average of all of them as a rank-one direction representative of the set.[1] However, given that the input directions, form a linearly independent set, the angles depend on the chosen state-space basis. For $m \leq n$ linearly independent input directions, where n is the dimension of the state space, state transformations may be found that make the pairwise angles between the directions arbitrarily close to 0 or 90 deg. It is, therefore, not clear whether the rank of the input map should be 1 or m or something in between.

A second approach is to group the input directions into a single multi-input mapping and to consider the singular values of this map.[2,3] If B_i is a linear system input map and $B = [B_1, \ldots, B_m]$ is an aggregate mapping, a reduced rank map is formed by combining the left singular vectors associated with the largest singular values of B, that is, once a threshold is chosen for what is to be considered a small singular value. This approach amounts to the first because the left singular vectors and singular values of B depend on the basis chosen for the state space unless, of course, only unitary transformations on the state space are allowed.

An underlying difficulty with both approaches is that the dynamics and input–output characteristics of the system are neglected. Consider, for example, a disturbance direction identification problem. A worst-case disturbance often is measured in an ℓ_2 sense, which suggests an $\mathcal{H}_{\infty}$ norm as the important system property to be maximized when performing a model approximation. On the other hand, in a control-blending problem, controllability and observability are the important system properties. A Hankel norm provides such an indicator as the square root of the largest eigenvalue of the product of controllability and observability gramians. Thus, the choice of the cost used in performing the model input reduction depends on the role the input plays in the system dynamics.

The following section introduces some notation. Section III presents an approach to model input reduction based on maximizing a system $\mathcal{H}_{\infty}$ norm. Section IV presents an approach based on maximizing a system Hankel norm. Section V illustrates an application to a disturbance direction identification problem; a numerical example is included. Section VI illustrates an application to a control-blending problem; a numerical example is also included.

II. Notation

Let G be a linear time-invariant system with k multidimensional inputs:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B_1\boldsymbol{u}_1 + \cdots + B_k\boldsymbol{u}_k, \qquad \boldsymbol{y} = C\boldsymbol{x}$$

The system has n states and p outputs and the k inputs $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ each have dimension m_i. It is also useful to define an aggregate-input mapping $B = [B_1, \ldots, B_k]$ that operates on an m-dimensional vector $\boldsymbol{u}$ where $m = \sum m_i$. Thus, if the inputs belong to real vector spaces as in $\boldsymbol{u}_i \in \mathcal{U}_i$ and $\boldsymbol{u} \in \mathcal{U}$, then the $\mathcal{U}_i$ form an orthogonal decomposition of $\mathcal{U}$ as in $\mathcal{U} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$. A single-input system derived from G is indicated by G_q and defined as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{q}\bar{u}, \qquad \boldsymbol{y} = C\boldsymbol{x}$$

where $\bar{u}$ is a real scalar, $\boldsymbol{q} : \mathcal{R} \mapsto \mathcal{R}^m$ and $\|\boldsymbol{q}\| = 1$. Finally, L_c and L_{cq} are the controllability gramians of the pairs (A, B) and $(A, B\boldsymbol{q})$. L_o is the observability gramian of the pair (C, A).

III. $\mathcal{H}_{\infty}$ Norm Maximization Approach

In a disturbance modeling problem, the worst-case disturbance direction, in an ℓ_2 sense, is indicated by maximizing the system $\mathcal{H}_{\infty}$ norm. The $\mathcal{H}_{\infty}$ norm of the full-order system (C, A, B) may be regarded as a search over frequency for the largest singular value of the frequency response matrix:

$$\|G\|_{\infty} = \max_{\omega} \left\{\sigma_{\max}[C(\jmath\omega I - A)^{-1}B]\right\} \tag{1}$$

The search may be tuned by applying frequency-dependent weights to the input and output. Form a model input reduction problem by finding q such that the $\mathcal{H}_{\infty}$ norm of the system with reduced-order

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input is maximized. The right singular vector associated with the largest singular value directly provides an input direction q such that

$$\|G_q\|_\infty = \|G\|_\infty$$

which must be a maximizing value.

An input map of any reduced order is composed by defining a projector $P_{m-1} = [I - \boldsymbol{q}(\boldsymbol{q}^T\boldsymbol{q})^{-1}\boldsymbol{q}^T]$ and a reduced-input map $B_{m-1} = BP_{m-1}$. Performing a model input order reduction on B_{m-1} will produce a next most important input direction and the process may be repeated. An appropriate order for the reduced-order input is found by inspecting the $\mathcal{H}_\infty$ norm of the system at each iteration with its most important directions removed.

In the control-blending problem, controllability and observability are important system features to be preserved. Maximizing the Hankel norm rather than the $\mathcal{H}_\infty$ norm is indicated for these types of problems. An approach to maximizing the system Hankel norm is developed in the next section.

IV. Hankel Norm Maximization Approach

Section IV.A describes without motivation an algorithm for Hankel-norm maximization—an algorithm suitable for control-blending problems. Development of the proposed algorithm is set forth in the following sections. Section IV.B considers only the system output dynamic characteristics, that is, the pair (C, A), in forming a model input-reduction problem. Section IV.C considers only the system input dynamic characteristics, the pair (A, B). Section IV.D considers both system input and output characteristics in the triple (C, A, B) and a Hankel-norm cost function is formed.

A. Algorithm for Hankel-Norm Maximization

1) Solve an eigenvalue problem for the state direction x_0:

$$L_o L_c \boldsymbol{x}_0 = \lambda_{\max}\boldsymbol{x}_0$$

Scale $\boldsymbol{x}_0$ so that $\|\boldsymbol{x}_0\| = 1$.

2) Find the solution to

$$X = \int_{-\infty}^{0} e^{-A^T\tau}\boldsymbol{x}_0\boldsymbol{x}_0^T e^{-A\tau}\,\mathrm{d}\tau$$

as the solution to the steady-state Lyapunov equation

$$0 = A^T X + XA + \boldsymbol{x}_0\boldsymbol{x}_0^T$$

3) Solve the eigenvalue problem, given later by Eq. (9b),

$$B^T X B\boldsymbol{q} = \lambda_{\max_q}\boldsymbol{q}$$

for the input-reduction vector $\boldsymbol{q}$. Scale $\boldsymbol{q}$ so that $\|\boldsymbol{q}\| = 1$.

4) Find the reduced-input controllability gramian L_{cq}

$$L_{cq} = \int_{-\infty}^{0} e^{-A\tau}B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\,\mathrm{d}\tau$$

as the solution to the steady-state Lyapunov equation

$$0 = AL_{cq} + L_{cq}A^T + B\boldsymbol{q}\boldsymbol{q}^T B^T$$

5) Solve the eigenvalue problem given later by Eq. (9a),

$$L_o L_{cq}\boldsymbol{x}_0 = \lambda_{\max_x}\boldsymbol{x}_0$$

for the state direction $\boldsymbol{x}_0$. Scale $\boldsymbol{x}_0$ so that $\|\boldsymbol{x}_0\| = 1$.

6) Convergence is achieved when $\lambda_{\max_x}$, which is the square of the Hankel norm of $(C, A, B\boldsymbol{q})$, changes insignificantly. Go back to step 2 if needed.

Development of the proposed algorithm is set forth in the following sections.

B. System Observability and Model Input Reduction

Consider the output of the system $\boldsymbol{y}(t)$ for $t \in [0, \infty)$ generated only by a state initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$. The ℓ_2 norm of $\boldsymbol{y}(t)$ is given by

$$\|\boldsymbol{y}\|^2_{\ell_2[0,\infty)} = \int_0^\infty \boldsymbol{x}_0^T e^{A^T\tau}C^T C e^{A\tau}\boldsymbol{x}_0\,\mathrm{d}\tau = \boldsymbol{x}_0^T L_o \boldsymbol{x}_0$$

where L_o, the observability gramian, is found as a solution to a steady-state Lyapunov equation

$$0 = A^T L_o + L_o A + C^T C$$

Define a reduced-order input map as $\bar{B} = B\boldsymbol{q}$, where $\boldsymbol{q} : \mathcal{R} \mapsto \mathcal{R}^m$ and $\|\boldsymbol{q}\| = 1$; that is, a dimension m input is to be reduced to a dimension 1 input. If the state initial condition $\boldsymbol{x}_0$ is restricted to Im B as $\boldsymbol{x}_0 = B\boldsymbol{q}$, a model input-reduction problem is to find $\boldsymbol{q}$ that maximizes $\|\boldsymbol{y}\|$:

$$\max_q J^o = \max_q \boldsymbol{q}^T B^T \int_0^\infty e^{A^T\tau}C^T C e^{A\tau}\,\mathrm{d}\tau\, B\boldsymbol{q} = \max_q \boldsymbol{q}^T B^T L_o B\boldsymbol{q} \tag{2}$$

This is solved as an eigenvalue problem:

$$J^o_{\max} = \lambda_{\max}\left(B^T L_o B\right)$$

with $\boldsymbol{q}$ taken to be the associated eigenvector. Because only the observability of the system pair (C, A) is considered, no weighting is given to the control energy required to reach the initial state $\boldsymbol{x}_0$.

C. System Controllability and Model Input Reduction

Having found an initial state direction $\boldsymbol{x}_0 \in$ Im B, note that the control energy required to reach $\boldsymbol{x}_0$ by applying a control $\omega(t)$ over $t \in (-\infty, 0]$ could be arbitrarily large. A well-known result[4] is that, for a given reachable state $\boldsymbol{x}_0$, the smallest signal $\omega \in \ell_2(-\infty, 0]$ that produces x_0 has a norm given by

$$\inf_\omega \left\{\|\boldsymbol{\omega}\|^2 | \boldsymbol{x}(0) = \boldsymbol{x}_0\right\} = \boldsymbol{x}_0^T L_c^{-1}\boldsymbol{x}_0$$

where L_c is the controllability gramian,

$$L_c = \int_{-\infty}^{0} e^{-A\tau}BB^T e^{-A^T\tau}\,\mathrm{d}\tau$$

also found as a solution to a steady-state Lyapunov equation:

$$0 = AL_c + L_c A^T + BB^T$$

A best initial state $\boldsymbol{x}_0$ could be considered to be one requiring the smallest signal $\boldsymbol{\omega} \in \ell_2(-\infty, 0]$. This is an eigenvalue problem because

$$\max_{x_0}\sup_\omega \frac{\|\boldsymbol{x}_0\|^2_{R^n}}{\|\boldsymbol{\omega}\|^2_{\ell_2(-\infty,0]}} = \max_{x_0}\frac{\boldsymbol{x}_0^T\boldsymbol{x}_0}{\boldsymbol{x}_0^T L_c^{-1}\boldsymbol{x}_0} = \lambda_{\max}(L_c)$$

where $\lambda_{\max}(L_c)$ is the largest eigenvalue of L_c and $\boldsymbol{x}_0$ is the associated eigenvector:

$$L_c\boldsymbol{x}_0 = \lambda_{\max}\boldsymbol{x}_0$$

A model input-reduction problem that follows is to find $\boldsymbol{q}$ with $\|\boldsymbol{q}\| = 1$ and the reduced-input map $\bar{B} = B\boldsymbol{q}$ by solving the following maximization problem:

$$J^c_{\max} = \max_q \max_{x_0} \frac{\boldsymbol{x}_0^T\boldsymbol{x}_0}{\boldsymbol{x}_0^T\left[\int_{-\infty}^0 e^{-A\tau}B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\,\mathrm{d}\tau\right]^{-1}\boldsymbol{x}_0}$$

$$= \max_q \max_{x_0} \frac{\boldsymbol{x}_0^T\int_{-\infty}^0 e^{-A\tau}B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\,\mathrm{d}\tau\boldsymbol{x}_0}{\boldsymbol{x}_0^T\boldsymbol{x}_0} \tag{3}$$

subject to $\|\boldsymbol{q}\| = 1$, which is equivalent to

$$J^c_{\max} = \max_q \max_{x_0} \boldsymbol{x}_0^T \int_{-\infty}^0 e^{-A\tau} B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\, d\tau \boldsymbol{x}_0 \tag{4}$$

subject to $\|\boldsymbol{q}\| = 1$ and $\|\boldsymbol{x}_0\| = 1$. Problem (4) is solved by adjoining two constraints to the cost using Lagrange multipliers; for example,

$$J^c_{\max} = \max_q \max_{x_0} \boldsymbol{x}_0^T \int_{-\infty}^0 e^{-A\tau} B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\, d\tau \boldsymbol{x}_0 - \lambda_x\left(\boldsymbol{x}_0^T\boldsymbol{x}_0 - 1\right) - \lambda_q(\boldsymbol{q}^T\boldsymbol{q} - 1)$$

The first-order variation of J^c with respect to $\boldsymbol{x}_0$ and $\boldsymbol{q}$,

$$\delta J^c = \left(\boldsymbol{x}_0^T \int_{-\infty}^0 e^{-A\tau} B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\, d\tau - \lambda_x \boldsymbol{x}_0^T\right)\delta\boldsymbol{x}_0 + \left(q^T B^T \int_{-\infty}^0 e^{-A^T\tau}\boldsymbol{x}_0\boldsymbol{x}_0^T e^{-A\tau}\, d\tau B - \lambda_q \boldsymbol{q}^T\right)\delta\boldsymbol{q}$$

provides a pair of eigenvalue problems as necessary conditions for J^c maximization:

$$\int_{-\infty}^0 e^{-A\tau} B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\, d\tau \boldsymbol{x}_0 = \lambda_x \boldsymbol{x}_0 \tag{5a}$$

$$B^T \int_{-\infty}^0 e^{-A^T\tau}\boldsymbol{x}_0\boldsymbol{x}_0^T e^{-A\tau}\, d\tau B\boldsymbol{q} = \lambda_q \boldsymbol{q} \tag{5b}$$

Furthermore, because $\|\boldsymbol{q}\| = \|\boldsymbol{x}_0\| = 1$, it follows that $\lambda_x = \lambda_q = J^c$. Thus, $J^c_{\max}$ is the largest eigenvalue of Eqs. (5) and the necessary condition for maximizing J^c is

$$\int_{-\infty}^0 e^{-A\tau} B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\, d\tau \boldsymbol{x}_0 = \lambda_{\max_x} \boldsymbol{x}_0 \tag{6a}$$

$$B^T \int_{-\infty}^0 e^{-A^T\tau}\boldsymbol{x}_0\boldsymbol{x}_0^T e^{-A\tau}\, d\tau B\boldsymbol{q} = \lambda_{\max_q} \boldsymbol{q} \tag{6b}$$

There is no known closed-form solution to Eqs. (6), but a convergent iterative solution, which is similar to D–K iteration for solving structured singular value problems,[5,6] is to solve Eq. (6a) for $\boldsymbol{x}_0$ while holding $\boldsymbol{q}$ fixed, then to solve Eq. (6b) for $\boldsymbol{q}$ while holding $\boldsymbol{x}_0$ fixed and repeat. The convergence of the proposed iteration can be shown as follows. First, the cost (6), either $J^c_k = \lambda_{\max_{x,k}}$ or $J^c_k = \lambda_{\max_{q,k}}$, increases monotonically with each iteration because the eigenvalue problems (6) may be expressed as maximization problems. Now, holding $\boldsymbol{q}$ fixed, Eq. (6a) gives $\lambda_{\max_x}$ for iteration k as follows:

$$\lambda_{\max_{x,k}} = \max_{x_0} \frac{x_0^T \int_{-\infty}^0 e^{-A\tau} B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\, d\tau \boldsymbol{x}_0}{\boldsymbol{x}_0^T\boldsymbol{x}_0} \tag{7}$$

Then, holding $\boldsymbol{x}_0$ fixed, Eq. (6b) gives $\lambda_{\max_q}$ for iteration $k+1$ as follows:

$$\lambda_{\max_{q,k+1}} = \max_q \frac{q^T B^T \int_{-\infty}^0 e^{-A^T\tau}\boldsymbol{x}_0\boldsymbol{x}_0^T e^{-A\tau}\, d\tau B\boldsymbol{q}}{\boldsymbol{x}_0^T\boldsymbol{x}_0} = \max_q \frac{\boldsymbol{x}_0^T \int_{-\infty}^0 e^{-A\tau} B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\, d\tau \boldsymbol{x}_0}{\boldsymbol{x}_0^T\boldsymbol{x}_0} \tag{8}$$

Therefore, $\lambda_{\max_{q,k+1}} \geq \lambda_{\max_{x,k}} \geq \lambda_{\max_{q,k-1}} \geq \cdots \geq \lambda_{\max_{x,1}}$. As long as $\|\boldsymbol{q}\| = \|\boldsymbol{x}_0\| = 1$, the cost J^c_k is bounded from above. Thus, J^c_k approaches a limit and the iteration must converge:

$$\lim_{k\to\infty} J^c_k = J^c_{\max}$$

With either $\boldsymbol{x}_0$ or $\boldsymbol{q}$ fixed, the global maximum in the other variable can be found by solving the eigenvalue problem. Although the joint maximization of $\boldsymbol{x}_0$ and $\boldsymbol{q}$ is not convex and the global convergence is not guaranteed, the experience of the authors has shown that in several practical applications the limit achieved is close to the upper bound given by $\lambda_{\max}(L_c)$ of the full-input system.

This approach is a dual to that of Sec. IV.B in the sense that only the controllability of the system pair (A, B) is considered. No weight is given to the energy required to reconstruct the initial state x_0 from the output pair (C, A).

D. System Input/Output Properties and Model Input Reduction

Both input, Eq. (2), and output, Eq. (3), considerations are combined by maximizing the Hankel norm of the system triple $G \stackrel{\Delta}{=} (C, A, B\boldsymbol{q})$ as follows:

$$J_{\max} = \max_q \sup_\omega \frac{\|\boldsymbol{y}\|^2_{\ell_2[0,\infty)}}{\|\boldsymbol{\omega}\|^2_{\ell_2(-\infty,0]}} = \max_q \max_{x_0} \frac{\boldsymbol{x}_0^T L_o \boldsymbol{x}_0}{\boldsymbol{x}_0^T\left[\int_{-\infty}^0 e^{-A\tau} B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\, d\tau\right]^{-1}\boldsymbol{x}_0}$$

Necessary conditions for this problem are a pair of eigenvalue problems similar to Eqs. (6) but with the premultiplication of an observability gramian:

$$L_o \int_{-\infty}^0 e^{-A\tau} B\boldsymbol{q}\boldsymbol{q}^T B^T e^{-A^T\tau}\, d\tau \boldsymbol{x}_0 = \lambda_{\max_x} \boldsymbol{x}_0 \tag{9a}$$

$$B^T \int_{-\infty}^0 e^{-A^T\tau}\boldsymbol{x}_0\boldsymbol{x}_0^T e^{-A\tau}\, d\tau B\boldsymbol{q} = \lambda_{\max_q} \boldsymbol{q} \tag{9b}$$

An iterative solution to Eqs. (9) similar to that for Eqs. (6) has been successfully applied.

V. Application to Disturbance Direction Estimation

Parametric uncertainties in a linear system model can often be modeled as scalar or low-dimensional unknown inputs. Given the disturbance direction, a disturbance decoupling or bounding controller may be constructed to provide robustness to variations in the unknown system parameters. In practice, identifying this direction has its own uncertainty in the sense that usually a set of directions is formed from which a representative direction is chosen. For the purpose of illustration, an observer-based method for identifying a disturbance direction is outlined in the following, taken from Ref. 1.

Consider a linear time-invariant system with a scalar disturbance d:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + Ed$$

The mapping E represents the direction of uncertainty acting on the system model. If this direction is not well known, it may be estimated by letting $\boldsymbol{w} = Ed$ and forming an observer to estimate $\boldsymbol{w}$. First, assume that $\boldsymbol{w}$ is a slowly time-varying vector of dimension n. Then append it to the state vector to form a $2n$-dimensional system,

$$\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{w}}\end{bmatrix} = \begin{bmatrix}A & I\\ 0 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{w}\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}\boldsymbol{u} \tag{10a}$$

Full-state information is assumed because Eq. (10a) is a system model and $\boldsymbol{x}$ is known perfectly:

$$y = \begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{w}\end{bmatrix} \tag{10b}$$

An observer based on model (10) is used to estimate $\boldsymbol{w}$. The measurements provided to the observer are from the full higher-order or nonlinear system or simulation. It is assumed that constant controls $\boldsymbol{u}$ are applied so that the observer eventually reaches a steady state and an estimate for $\boldsymbol{w}$; that is, $\hat{\boldsymbol{w}}$ becomes a constant. The disturbance direction E is a normalized $\hat{\boldsymbol{w}}$. In general, the direction E found will depend on the size and direction of the control vector $\boldsymbol{u}$ applied to the nonlinear plant. Thus, a system model with several disturbance directions is formed as follows:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + E_1 d_1 + E_2 d_2 + \cdots + E_m d_m$$

The problem now is to determine an appropriate reduced dimension and range for the disturbance distribution matrix $[E_1, E_2, \ldots, E_m]$. As outlined in the Introduction, several approaches exist for constructing E from E_i. Because E is to represent an exogenous system influence, it is assumed that a worst-case disturbance is of the greatest interest. Hence, an $\mathcal{H}_\infty$ norm is maximized, rather than system controllability, with respect to E.

A numerical example follows from the longitudinal dynamics of an automobile linearized as the vehicle is driven on a straight and level road at a constant speed of 25 m/s. A small slope in the road appears as a disturbance in the dynamics. The states are engine manifold air mass, engine speed, longitudinal and vertical velocity, vertical position, the sum of two front wheel speeds, and the sum of two rear wheel speeds. The control inputs are throttle and brake pressure actuator commands. The disturbance directions $E = [E_1, E_2, E_3, E_4]$ are found for road slopes of -1, -2, -3, and 0 deg, respectively, when the brake pressure is held at the trim value and the throttle is fixed to a value such that the vehicle speed is 27 m/s. Data for the system are as follows:

$$A = \begin{bmatrix} -0.0521 & -0.2213 & 0.2681 & -0.0121 & 0.0136 & 0.0084 & -0.0078 \\ -0.3007 & -8.0277 & 19.0734 & -1.1013 & 0.0795 & 0.2471 & 0.0378 \\ -0.3263 & -19.7571 & -51.0638 & -3.2675 & -4.8766 & -2.4258 & 0.0040 \\ 0.0454 & 2.4036 & 15.7922 & -2.1857 & 6.4655 & -0.2062 & 0.0495 \\ 0.0219 & 1.1136 & 8.6428 & -7.1817 & -0.6526 & -0.2171 & 0.9316 \\ 0.0116 & 0.5928 & 3.8335 & -1.0926 & -0.6513 & -0.9851 & 5.9628 \\ 0.0154 & 0.7868 & 4.8494 & -1.4900 & -1.0329 & -6.5688 & -2.5996 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.0362 & 0.0711 & 0.0976 & 0.0164 \\ 0.0786 & 0.1585 & 0.2222 & 0.0343 \\ 0.2191 & 0.3765 & 0.4450 & 0.1128 \\ 0.0404 & 0.0452 & 0.0628 & 0.0419 \\ 0.1057 & 0.1292 & 0.1559 & 0.0943 \\ 0.3199 & 0.2969 & 0.2822 & 0.3294 \\ -0.9107 & -0.8492 & -0.7970 & -0.9310 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.0075 & 0.4605 & 0.3710 & 0.1023 & 0.0513 & 0.0340 & -0.0137 \\ 0.7318 & 2.7938 & -2.8640 & 0.1680 & -0.0415 & -0.0491 & -0.0029 \\ 0.0028 & 0.1711 & -0.2654 & 0.0765 & -0.0161 & 0.0093 & -0.0008 \\ 0.0000 & -0.0007 & -0.0005 & -0.0216 & -0.0496 & -0.0438 & 0.0697 \\ -0.0000 & -0.0024 & 0.0050 & 0.0111 & 0.0205 & -0.0027 & 0.0009 \\ 0.4214 & -0.1440 & 0.0371 & 0.2203 & -0.1764 & -0.0129 & 0.1051 \\ 0.4211 & 0.1318 & -0.4410 & -0.2741 & -0.0304 & -0.0734 & 0.0585 \end{bmatrix}$$

Because these inputs are system disturbances, an $\mathcal{H}_\infty$ norm is used to find the worst-case reduced-order input:

$$\boldsymbol{q} = [0.2830 \quad 0.5591 \quad 0.7689 \quad 0.1272]^T$$

The $\mathcal{H}_\infty$ norm of the full- and reduced-order input systems is 2.3959.

By comparison, for this example, the iterative approach of Sec. IV.A produces nearly the same result after two iterations. Furthermore, the Hankel norms of both full- and reduced-order input systems are nearly the same: 1.3432. Considering the Hankel norm and system eigenvectors, the full- and reduced-input system input-output properties appear to be similar. This might seem surprising because the four disturbance directions are, pairwise, 24.26, 17.91, 17.52, 11.08, 6.85, and 6.64 deg apart.

VI. Control-Blending Example

In this section, a numerical example for the control-blending problem is presented. As discussed in the Introduction, preservation of the system controllability and observability properties is important in this type of problem, so a Hankel-norm maximization approach is preferred.

The lateral dynamics of an F-16 MATV aircraft are given in Ref. 3 as a fifth-order linear model with four control inputs. The flight condition is for Mach 0.375, 40,000 ft altitude, and 45 deg angle of attack:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \qquad \boldsymbol{y} = C\boldsymbol{x}$$

The states are roll rate, yaw rate, side slip, bank angle, and a state associated with the computation of one of two lateral control variables. See Ref. 3 for details on the definition of the control variables. The controls are differential tail, differential wing flaps, rudder, and lateral thrust vectoring. The outputs that are the most important for control design are the roll rate and yaw rate. Data for the system are as follows:

$$A = \begin{bmatrix} -0.7437 & 0.5440 & -2.7490 & 0 & 0 \\ 0.1584 & -0.2160 & -1.1490 & 0 & 0 \\ 0.7070 & -0.7070 & -0.0170 & 0.0875 & 0 \\ 1.0000 & 0.1386 & 0 & 0 & 0 \\ 0.7071 & -0.7071 & -0.0075 & 0.0707 & -0.1000 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.8153 & -0.0004 & 0.3234 & -0.0047 \\ -0.1595 & -0.0664 & -0.1655 & -0.8330 \\ 0 & 0 & 0.0028 & 0.0125 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4118 & 1.8240 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Before applying any input-order reduction scheme, the inputs are normalized by scaling them by the inverse of their maximum values. The maximum values are given in the diagonal matrix S:

$$S = \mathrm{diag}\{50, 10, 30, 15\}$$

Because the linear model is to be used for feedback control design, the reduced-order input is found by maximizing a Hankel norm.

A first-order reduced input is found in three iterations of Eqs. (9). The blending matrix scaled by S and reduced-order system Hankel norm are

$$S \cdot \boldsymbol{q}_1 = [43.51 \quad 0.2043 \quad 9.097 \quad 5.817]^T, \qquad J = 257362$$

Using the largest singular value approach described in the Introduction and used in Ref. 3 produces the following blending matrix, again scaled by S, and smaller Hankel norm:

$$S \cdot \boldsymbol{q}_{\mathrm{sv}} = [-48.35 \quad -0.03424 \quad -7.391 \quad -0.9671]^T$$

$$J = 228080$$

To allow for control-variable decoupling, a second input direction is found. The scaled blending matrix and system Hankel norm are

$$S \cdot \boldsymbol{q}_2 = [22.01 \quad -0.4416 \quad -3.735 \quad -13.32]^T, \qquad J = 257802$$

For comparison, the full-input Hankel norm is 257,802. Note that the four scaled control input directions are, pairwise, 79.25, 50.52, 85.56, 72.66, 65.45, and 36.30 deg apart.

VII. Conclusions

A reduced-input system problem is formed with the objective of preserving system input–output properties. For exogenous inputs, a reduced-order input system $\mathcal{H}_\infty$ norm is maximized and an analytic solution is given. For control inputs, a reduced-order input system Hankel norm is maximized. Necessary conditions for the Hankel norm maximization are stated and an iterative solution is proposed. Although global convergence cannot be guaranteed, many practical applications have shown the achieved maxima to be very close to a known upper bound.

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Nonlinear Modeling of Spacecraft Relative Motion in the Configuration Space

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I. Introduction

THE analysis of relative spacecraft motion constitutes an issue of increasing interest. It was in the early 1960s that Clohessy and Wiltshire (CW) first published their celebrated work that utilized a Hill-like rotating coordinate system to derive expressions for the relative motion between satellites.[1] The CW linear formulation assumed small deviations from a circular reference orbit and used the initial conditions as the constants of motion. Since then, recognizing some of the limitations of this approach, others have generalized the CW equations for eccentric reference orbits[2] and to include perturbed dynamics.[3,4]

An important modification of the CW linear solution is the use of orbital elements as constants of motion instead of the Cartesian initial conditions. This concept, originally suggested by Hill,[5] has been widely used in the analysis of relative spacecraft motion.[3,6,7] Using this approach allows the examination of the effect of orbital perturbations on the relative motion via variational equations such as Lagrange's planetary equations (LPEs) or Gauss's variational equations (GVEs). Moreover, utilizing orbital elements facilitates the derivation of high-order, nonlinear extensions to the CW solution.

There have been a few reported efforts to obtain high-order solutions to the relative motion problem. Recently, Karlgaard and Lutze proposed formulating the relative motion in spherical coordinates in order to derive second-order expressions.[8] The use of Delaunay elements has also been proposed. For instance, Alfriend et al. derived differential equations in order to incorporate perturbations and high-order nonlinear effects into the modeling of relative dynamics.[9]

The present work establishes a methodology to obtain arbitrary high-order approximations to the relative motion between spacecraft by utilizing the Cartesian configuration space in conjunction with classical orbital elements. In other words, we propose utilizing the known inertial expressions describing vehicles flying in elliptic orbits in order to obtain, using a Taylor-series approximation, a time-series representation of the motion in a rotating frame, where the coefficients of the time series are functions of the orbital elements. We subsequently show that under certain conditions, this time series becomes a Fourier series. More important, in the process of the derivation, there is no need to solve differential equations. This significant merit results directly from utilization of the known inertial configuration space. The high-order approximation we present also provides important insights into boundedness of relative formation dynamics.

II. Problem Formulation

In the procedure to follow we study the relative motion of N vehicles (termed follower spacecraft) in arbitrary elliptic orbits relative to a circular reference orbit. We utilize the following standard coordinate systems: $\mathcal{I}$, a geocentric-equatorial inertial frame; $\mathcal{P}$, a

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